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AN EXTENSION OF AN ALGORITHM DEVELOPED BY P. HAMMER AND S. RUDEANU FOR MINIMIZING FRACTIONAL PSEUDO-BOOLEAN FUNCTIONS

INTRODUCTION AND THE ALGORITHM OF INTEREST

Several methods (including the ones in reference) have been developed in order to minimize (or maximize) a fractional pseudo-boolean function of the form

$$f(x) = \frac{a_0 + a_1 x_1 + \ldots + a_k x_k}{b_0 + b_1 x_1 + \ldots + b_{k} x_k}, \quad x_j \in \{0,1\} \qquad (0.1)$$

with or without constraints. In these methods either the positivity or negativity of the coefficients a_j and b_j is particularly observed.

P. Hammer and S. Rudeanu[1,2] developed an algorithm, called «Algorithm-I» in order to achieve all vectors $x \in \{0,1\}^k$ minimizing the function 7 with out constraints, under the assumption the coefficients a_j and b_j ($j=0, 1, 2, \ldots, k$) to be non-negative, but the strict positivity of the constant term b_0 is particularly requested.

This algorithm, based on three lemmas, may be briefly described as follows:

ALGORITHM-I

Step 1. If, for some $j=1, 2, \ldots, k$, $a_j=o$ and $b_j=o$ put $x_j=1$. If $a_j > o$ and $b_j=o$ put $x_j=o$.

Step 2. Define the set I of all remaining indices with $a_j/b_j \leq a_o/b_o$ and the set J with $a_j/b_j > a_o/b_o$.

Step 3. For every $j \in J$ put $x_j=o$.

Step 4. Find the smalest index $j_1 \in I$ for which $a_{j_1}/b_{j_1} = \min_{j \in I}(a_j/b_j)$.

4a. If $a_{j_1}/b_{j_1} < a_o/b_o$, set $x_{j_1}=1$ and compute new constant terms $a'_o = a_o + a_{j_1}$ and $b'_o = b_o + b_{j_1}$.

4b. If $a_{j_1}/b_{j_1} = a_o/b_o$, set $x_{j_1}=p$, where p denotes a 0-1 parameter.

In both cases repeat the algorithm from the step 2 until all the variables x_j are exausted.

It is now clear the fact that after executing the step 1 a new function of the same form of 0.1 is resulted, in which all coefficients a_j and b_j are strictly positive. In the other hand, the resulted value of the fraction a_0/b_0 is of essential importance since it plays a critical role in all next repeated steps 2 to 4. It is also clear that if this fraction takes a specific value in starting the algorithm-I from the step 2, it will keep a specific value in all algorithm's repetitions as well. Thus the only point needed to be examined is the initial value of the rate a_o/b_o.

If $b_o>o$ the rate a_o/b_o accepts always a specific value and the algorithm-I leads to all minimizing points of the function f. But, what about $b_o=0$? This is the case in which the algorithm cannot be started and it is not considered[1, 2*].

In the present paper the lemma 1, stated by Hammer and Rudeanu[2], page 153, is going to be generalized in order to include the case $b_o=0$ as well as the necessary modifications to the algorithm-I that should be made in order to be applicable in all combinations of the zero values of a_o and b_o.

GENERALIZATION OF LEMMA-1 AND EXTENSION OF THE ALGORITHM-I

Let a_j and b_j be strict positive in the initial function 0.1, $j=1, 2, ..., k$. This assumption does not cause any loss to generality since if ($a_j=o$, $b_j>o$) or ($a_j>o$, $b_j=o$) we apply the step 1 and a new function is resulted with all coefficients a_j and b_j strict positive. Thus, the rates a_j/b_j pocess specific values for all $j=1, 2, ..., k$.

Instead of the rate a_o/b_o, we introduce a new critical quantity q in order to define the sets I and J in a new way, as follows:

i) If $b_o>o$ set $q=a_o/b_o$.

ii) If $b_o=o$ and $a_o=o$ set $q=\min\limits_{1\leq j\leq k}\{a_j/b_j\}$.

iii) If $b_o=o$ and $a_o>o$ set q equal to an arbitrary value greater than $\max\limits_{1\leq j\leq k}\{a_j/b_j\}$.

Now, the sets I and J are defined as follows:

$$I=\{j : a_j/b_j\leq q\} \tag{1.1}$$

$$J=\{j : a_j/b_j>q\} \tag{1.2}$$

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* In later papers, stated by P. Robillard[4] and M. Florian and P. Robillard[3], generalizations of this algorithm are derived but, the strict positivity of the coefficient b_o is still meant.

Under the conditions, stated above, the following generalization of lemma-1 is valid.

LEMMA-1*

If the sets I and J are defined as in 1.1 and 1.2 the relation

$$\frac{a_o + \sum_{j\varepsilon I} a_j x_j}{b_o + \sum_{j\varepsilon I} b_j x_j} < \frac{a_o + \sum_{j\varepsilon I} a_j x_j + \sum_{j\varepsilon J} a_j x_j}{b_o + \sum_{i\varepsilon I} b_j x_j + \sum_{i\varepsilon J} b_i x_j} \tag{1.3}$$

holds.

Proof. If $b_o > o$ then we are faced with the case studied[2]; let now $b_o = o$ and $a_o = o$. If j_0 is the index for which

$$\frac{a_{jo}}{b_{jo}} = \min_{1 \leq j \leq k} \left\{ \frac{a_j}{b_j} \right\} \tag{1.4}$$

then

$$q = \frac{a_{jo}}{b_{jo}} \tag{1.5}$$

and for all $j\varepsilon$ I the relations

$$\frac{a_j}{b_j} = \frac{a_{jo}}{b_{jo}} \quad \text{and} \quad \frac{a_{jo}}{b_{jo}} < \frac{a_j}{b_i} \tag{1.6}$$

hold which yield to

$$\frac{\sum_{j\varepsilon I} a_j x_j}{\sum_{j\varepsilon I} b_j x_j} = \frac{a_{jo}}{b_{jo}} \quad \text{and} \quad \frac{a_{jo}}{b_{jo}} < \frac{\sum_{j\varepsilon J} a_j x_j}{\sum_{j\varepsilon J} b_i x_i} \tag{1.7}$$

especially. The last two relations 1.7 combined lead to

$$\frac{\sum_{j\varepsilon I} a_j x_j}{\sum_{j\varepsilon I} b_j x_j} \leq \frac{\sum_{j\varepsilon I} a_j x_j + \sum_{j\varepsilon J} a_j x_j}{\sum_{i\varepsilon I} b_j x_j + \sum_{j\varepsilon J} b_j x_j} \tag{1.8}$$

in which relation the equality holds when $J=\emptyset$. The relation 1.8 means that the lemma-1* in the case $a_o = o$, $b_o = o$ is valied. Finally, let the case $b_o = o$, $a_o < o$. In this last case $J = \emptyset$ and the relation 1.3 clearly holds with respect the equality. Therefore, the lemma-1* is completely proved.

On the basis of the lemma-1*, proved above and the lemmas 2 and

3 stated in [2], the algorithm-I can be extended, in order to include all combinations between the zero values of the constant terms a_o and b_o, as follows:

ALGORITHM-I*

Step 1. If for some $j=1,2,\dots,k$ $a_j=o$ and $b_j>o$ set $x_j=1$.
If $a_j<o$ and $b_j=o$ set $x_j=o$.

Step 1*. If $b_o>o$ set $q=\frac{a_o}{b_o}$

If $b_o=o$ and $a_o=o$ set $q=\min\limits_{1\leq j\leq k}\left\{\frac{a_j}{b_j}\right\}$

If $b_o=o$ and $a_o>o$ set $q>\max\limits_{1\leq j\leq k}\left\{\frac{a_j}{b_j}\right\}$

Step 2*. Define the set I to be the set of all j with $\frac{a_j}{b_j}\leq q$ and the set J to be the set of all j with $\frac{a_j}{b_j}>q$.

Step 3. Set $x_j=o$ for every $j\in J$.

Step 4. Find the smallest index $j_1\in I$ for which $\frac{a_{j1}}{b_{j1}}=\min\limits_{j\in I}\left\{\frac{a_j}{b_j}\right\}$

4a. If $\frac{a_{j1}}{b_{j1}}<q$ set $x_{j1}=1$ and compute new constant terms $a'_o=a_o+a_{j1}$ and $b'_o=b_o+b_{j1}$.

4b. If $\frac{a_{j1}}{b_{j1}}=q$ set $x_{j1}=p$, where p is an 0-1 parameter and compute new constant term $a'_o=a_{j1}$ and $b_o=b_{j1}$.

In both cases repeat the algorithm from the step 1* until all variables x_j are exausted.

It is worth to note at this point that, when the algorithm-J* is applied to the case $a_o=b_o=o$, that is when we are faced with the minimization of the function

$$g(x)=\frac{a_1x_1+\dots+a_kx_k}{b_1x_1+\dots+b_kx_k}, \tag{1.9}$$

then every repetition of the algorithm-I* ends at the step 4a and so, this is an indirect way to achieve the very simple procedure:

PROCEDURE I

Step 1. Find the set $I = \left\{ j : \frac{a_j}{b_j} = \min_{1 \leq j \leq k} \{\frac{a_j}{b_j}\} \right\}$ and the set

$$J = \left\{ j : \frac{a_j}{b_j} > \min_{1 \leq j \leq k} \{ \frac{a_j}{b_j} \} \right\}.$$

Step 2. For every $j \in I$ set $x_j = p_j$, where p_j is a 0-1 parameter, while for every $j \in J$ set $x_j = o$.

It should be noted that the solution (0,...,0) will appear if we combine the values of the parameters p_j. Since this solution leads to a non-specified value of the function g it must be rejected from the set of solutions.

EXAMPLES

Example 2.1 (Algorithm I application)*

Let the function

$$f(x_1,x_2,x_3,x_4,x_5) = \frac{x_1 + 4x_2 + x_3 + 2x_4 + 5x_5}{2x_1 + 2x_3 + 4x_4 + x_5} \tag{2.1}$$

to be minimized. The step-by-step application of the algorithm I* is as follows:

1. Since $a_2 = 4 > 0$ and $b_2 = 0$ we set $x_2 = 0$ and the new function

$$f_1(x_1,x_2,x_3,x_4,x_5) = \frac{x_1 + x_3 + 2x_4 + 5x_5}{2x_1 + 2x_3 + 4x_4 + x_5} \tag{2.2}$$

is to be minimized. All its coefficients are strictly positive exept $a_o = b_o = o$.

1*. Since $a_o = b_o = o$ we choose $\frac{a_o}{b_o} = \min_{j \in S} \frac{a_j}{b_j} = \frac{1}{2}$, where S={1,3,4,5}

2. We find $I = \{1,3,4\}$ and $J = \{5\}$

3. We put $x_3 = o$

4. The smallest index $j_1 \in I$ for which $\frac{a_{j_1}}{b_{j_1}} = \frac{a_o}{b_o} = 1$ is $j_1 = 1$. Therefore we put $x_1 = p_1$. The new function is now

$$f_2(x_3,x_4) = \frac{x_3 + 2x_4}{2x_3 + 4x_4} \tag{2.3}$$

2'. $I' = \{3,4\}$, $J' = \emptyset$

3'. No $j \in J'$

4'. Now $j_1 = 3$ with $\frac{a_{j_1}}{b_{j_1}} = \frac{a_o}{b_o} = \frac{1}{2}$ and so $x_3 = p_3$. The new function is

$$f_3(x_4) = \frac{2x_4}{4x_4} \tag{2.4}$$

2''. $I'' = \{4\}$, $J'' = \emptyset$

3''. No $j \in J''$

4''. The final index is $j_1 = 4$ with $\frac{a_{j_1}}{b_{j_1}} = \frac{a_o}{b_o} = \frac{1}{2}$ and so $x_4 = p_4$.

The algorithm terminates at this point since all variables are exausted. Thus, the optimal solution is

$$\bar{x}^*(p_1,p_3,p_4) = (p_1,0,p_3,p_4,0) \tag{2.5}$$

under the condition p_1, p_3, p_4 to be not all equal to zero. The minimal value of f is

$$f(\bar{x}^*) = \frac{1}{2} \tag{2.6}$$

Example 2.2 (Procedure I application)

Let the function below to be minimized.

$$f(x_1,x_2,x_3,x_4,x_5) = \frac{2x_1 + 3x_2 + x_3 + 2x_4 + 5x_5}{2x_1 + 2x_2 + x_3 + x_4 + 3x_5} \tag{2.7}$$

In this function all coefficients a_i, b_i are strictly positive while $a_o = b_o = o$. Thus we may apply the procedure I.

1. Since $\min\limits_{1 \le j \le 5} \frac{a_j}{b_j} = 1$ the sets I and J are:

   $I = \{1,3\}$ and $J = \{2,4,5\}$

2. Thus $x_1 = p_1$, $x_3 = p_3$ and $x_2 = x_4 = x_5 = 0$.

   The solutions of the problem min f are

   $x^*(p_1,p_3) = (p_1,0,p_3,0,0)$

and the minimum value of f is

$f(x^*) = 1$;

REFERENCE

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