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SOLUTIONS OF THE DYNAMIC INPUT-OUTPUT MODEL: A CRITICAL REVIEW

PLAN

ABSTRACT

The main purpose of this paper is to provide a critical review of the alternative solutions of the dynamic Input-Output model; the advantages and disadvantages of each one; and to present a modified model based on an econometric approach. The discussion of the alternative solutions indicates that the use of the forward-in-time-integration approach is necessary since it is based on initial (actual) information. The problem with the backward-in-time-integration approach is that it does not prevent the determination of initial (actual) year output level and it is likely to result in inconsistent projections. On the other hand, the main problem of the modified model is data collection, which restricts its use in applied analysis.

I. INTRODUCTION

In the static input-output model we are interested in interindustry flows among sectors. Each input-output table represents how much of the output of one sector is used as an input by all other sectors and how much goes to final demand. In addition, some of these sectors produce capital goods which are used by other sectors as factors of production. Some sectors may, however, produce both goods; that is, consumable and capital goods. In order to examine these capital stock flows, a special kind of a model is needed; namely, a dynamic input-output model.

In the dynamic input-output model, as in the static, all components of the final demand are exogenously determined and connected into technical and capital coefficients with each year's level of output. The functioning of this model is similar to that of the static if it used to forecast output over time for a given level of final demand. Nevertheless, the assumption of constant technical and capital coefficients over time restricts the projection capability of the model since a constant level technology is implied. Although non-realism, this assumption is necessary since Leontief production functions are used, which assume no substitution between factors of production.

The forecasting procedure of the dynamic input-output model is based on either initial or terminal conditions. In both cases the solution entails forecasts for different output levels over time. It is obvious though that the solution based on initial conditions is more realistic than the one based on terminal conditions. This prevails because the first utilizes actual information (initial conditions) whereas the latter utilizes hypothetical (terminal) values.

The main objective of this paper is to present the alternative solutions of the dynamic input-output model; the advantages and disadvantages of each one; and to develop a modified dynamic input-output model based on an econometric approach. The rest of this paper is organized as follows: in section II, a detailed discussion on the construction of the dynamic input-output model is presented. Backward- and forward-in-time-integration solutions of the dynamic input-output model are developed in sections III and IV, respectively. In the fifth section, the modified Leontief dynamic model is presented. Concluding remarks follow.

II. CONSTRUCTING THE DYNAMIC INPUT-OUTPUT MODEL

In the dynamic input-output model is presupposed the existence of capital stocks that are necessary for production; i.e., each industry keeps some capital stocks in order to be able to continue its production process next year. Hence, in the dynamic formulation, a matrix B is constructed in addition to matrix A. The elements of this matrix represent the stock of each commodity, that each sector must have in hand for each level of production as capital good. As a first step to estimate these coefficients, i^{th} sector's value of output needed as stock for the j^{th} sector should be measured. This denoted by U_{ij}. Then, a coefficient can be estimated by dividing U_{ij} by the j^{th} sector output. The economic interpretation of matrix B is quite obvious. In reality, when a production process takes place, capital goods having multi-period usage are used (i.e. machines and buildings) as well as goods exchanged between sectors at each particular period of time. Capital stocks for the j^{th} sector may consist of fixed investment items such as buildings and machines and also of goods bought by the particular sector as inventories to be used later as inputs. All the above items are included in U_{ij}.

The elements of matrix B, $b_{ij} - U_{ij}/X_j$, represent i^{th} sector's output used by j^{th} sector as capital stock (in money terms) in order the j^{th} sector to be able to produce output of worth equal to one unit of country's currency. The elements of matrix B as well as those of matrix A are assumed to be known and time-invariant. This simply means that there are no any changes in flows of goods between sectors and in addition, each sector uses the same level of factors in every year's production.

An initial level of machinery buildings and other capital goods is needed for current production to take place. However, if an economy is growing, anticipated (next year) production is different from current (this year) production and thus, the amount of supporting capital may change. According to the above assumptions of time-invariant elements of matrices A and B, it must be assumed that if an industry increases its rate of production during a period of time, it will simultaneously increase its stock of goods such that the ratio of its capital goods stock and its rate of output growth remains constant.

An often utilized assumption is that the amount of additional capital stocks that the j^{th} sector obtains from the i^{th} sector at time period t+1 is given by $b_{ij}(X_{jt+1} - X_{jt})$. That is, the amount of i^{th} sector's production necessary to satisfy the additional demand by the j^{th} sector for capital stock for next year's production is given by the observed capital coefficient, b_{ij}, multiplied by the change in j^{th} sector's output between this and next year. Using capital coefficient in such a way, it is implicitly assumed that j^{th} sector's production reaches its frontier of capacity efficiency, since the anticipated increase in production requires new capital goods. In such a case, the change in j^{th} sector's output between this and the next year, $X_{jt+1} - X_{jt}$, is assumed to be positive. However, $X_{jt+1} - X_{jt}$ could also be negative or zero. In general, we are usually concern with the sectorial consequences of economic growth; that is, with the case in which this term is strictly positive.

In the dynamic input-output model, the typical equation for the output of i^{th} sector in period t is:

$$X_{it} = \sum_{j=1}^{n} a_{ij} X_{jt} + \sum_{j=1}^{n} b_{ij} \left(X_{jt+1} - X_{jt} \right) + Y_{it}$$

or in matrix notation:

$$X_t = AX_t + B(X_{t+1} - X_t) + Y_t \tag{1}$$

where AX_t denotes intermediate demand for goods by industries; $B(X_{t+1} - X_t)$ gives the distribution of inputs to investment alternatives; Y_t is the vector of final demand exogenously determined in period t; and X_t is the vector of output level in period t. From equation (1), we have:

$$(I - A + B)X_t - BX_{t+1} = Y_t \tag{2}$$

or by rearranging terms:

$$(I - A + B)X_t - Y_t = BX_{t+1} \tag{3}$$

If the time superscripts denote years, equation (2) represents a set of relationships between gross output and final demand starting now (year t=0) and extending t years into the future.

By measuring capital coefficients, it is possible to distinguish between replacement and expansion capital. Replacement capital is, for example, investment for replacing depreciated equipments and it is a function of current production, X_t. Expansion capital, on the other hand, is investment in new equipments to expand production capacity. It is a function of industry growth and it is determined by the difference between current and past production. In such a case, equation (2) can be written as:

$$(I - A - D + B)X_t - BX_{t+1} = Y_t$$

where D is a newly added matrix of replacement capital coefficients and B is now the matrix of expansion capital coefficients.

According to Miernyk (1970), for measuring the elements of matrix B it is necessary to isolate expansion investment in the base year (t=0) by sector. The matrix of capital coefficients provides the nexus between expansion investment and capacity on the one hand and output on the other. The relationship between expansion investment and capacity in the base period is held constant throughout the projection period. Then, expansion investment becomes an endogenous variable in the system and it is determined by the initial conditions and the exogenous forecasts of final demand. Each element of matrix B is the ratio of the amount of expansion investment required for a per unit change in capacity; that is, $b_{ij} = I_{exp} / DC$, where I_{exp} is the expansion part of investment;

and DC is the difference between required capacity in year t and capacity in year t–1.

As long as not all sectors produce significant amounts of capital goods, some rows of matrix B contain only zero elements. A typical example of this is the agricultural sector. In general, agricultural products are not used in the production of investment goods. A farmer uses a tractor (a capital good) but his output is almost wholly consumed. Thus, in a multi-sector model, only a small number of sectors contribute to capital formation. This can be lead to singularity problems with respect to matrix B. The rank of matrix B may be much smaller than n, which is the number of sectors. As Kendrick (1972) noted this is because there are multiple lags on both fixed and inventory investment and usually the lags of the former type of investment are longer than those of the latter.

A simple method to overcome the singularity problem is to aggregate the sectors producing significant capital goods with some of those producing investment goods. However, this method suffers from the typical problem of aggregate analysis since essential features of a multi-sectoral economy are not captured. Furthermore, the impacts of each sector's commodities on the national economy cannot be observed. Since a certain degree of disaggregation is desired in such models, the above mentioned method should not be used.

Equation (1) represents a system of simultaneous, linear, difference equations in which the variables (output of each sector) are related, via the coefficients in matrices A and B, with the final demand, for different periods of time. From equation (3):

$$RX_t - BX_{t+1} = Y \qquad (4)$$

where $R = [I - A + B]$ or in matrix form:

$$\begin{bmatrix} R & -B & 0 & \dots & 0 & 0 \\ 0 & R & -B & \dots & 0 & 0 \\ \dots & \dots & \dots & & \dots & \dots \\ \dots & \dots & \dots & & -B & 0 \\ 0 & 0 & 0 & \dots & R & -B \end{bmatrix} \begin{bmatrix} X_0 \\ \dots \\ \dots \\ X_t \\ X_{t+1} \end{bmatrix} = \begin{bmatrix} Y_0 \\ \dots \\ \dots \\ \dots \\ Y_t \end{bmatrix} \qquad (5)$$

For an n–sector economy, it is obvious from equation (4) that we have a system of difference equations with nt equations and (n+1) unknowns.

Thus, two problems arise; namely, possible singularity of matrix B and overidentification of the system of equations (5). A number of approaches to overcome the above problems has been developed. Each approach is based on different assumptions. A throughout discussion of these alternative approaches is presented in the next section.

III. BACKWARD-IN-TIME-INTEGRATION OR LEONTIEF SOLUTION OF THE DYNAMIC SYSTEM

Probably the simplest technique to overcome the singularity problem of matrix B and the overidentification of the system of equation (5) is that proposed by Leontief. He suggested a method which stresses the necessity of integrating iteratively backwards in time using the terminal conditions and, thus it overcomes the singularity problems by using the inverse of the matrix I–A+B, instead of the inverse of the matrix B. Leontief's method assumes that terminal values specify au desired characteristics of the system at the end of the period over which the model is employed. Assuming that what is happening after terminal time t does not matter, $X_{t+1}=0$. This assumption entails negatively of inputs to investment in the terminal year. This approach moves backward in time, starting at the end of the planning horizon and ended initial time t_0.

Since $X_{t+1}=0$ is eliminated from the vector X in equation (5), making the last column of the coefficient matrix in the same equation unnecessary. Equation (5) can be rewritten as:

$$\begin{bmatrix} R & -B & 0 & \dots & 0 & 0 \\ 0 & R & -B & \dots & 0 & 0 \\ \dots & \dots & \dots & & \dots & \dots \\ \dots & \dots & \dots & & 0 & -B \\ 0 & 0 & 0 & \dots & 0 & R \end{bmatrix} \begin{bmatrix} X_0 \\ \dots \\ \dots \\ \dots \\ X_t \end{bmatrix} = \begin{bmatrix} Y_0 \\ \dots \\ \dots \\ \dots \\ Y_t \end{bmatrix} \qquad (6)$$

Equation (6) represents a system of (nt) equations with (nt) unknowns. It can easily be solved for each year given final demand and provided that the inverse of matrix R exists. For example, for time t=3 and for a given current year final demand and exogenously determined final demands for the next three years, equation (6) is written as:

$$\begin{aligned} RX_0 - BX_1 &= Y_0 \\ RX_1 - BX_2 &= Y_1 \\ RX_2 - BX_3 &= Y_2 \\ RX_3 &= Y_3 \end{aligned}$$

or in matrix notation:

$$\begin{bmatrix} R & -B & 0 & 0 \\ 0 & R & -B & 0 \\ 0 & 0 & R & -B \\ 0 & 0 & 0 & R \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} Y_0 \\ Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} \qquad (7)$$

Equation (7) gives $X_3=R^{-1}Y_3$. Having X_3, we can find X_2 from the third equation in (7) as:

$$X_2 = R^{-1}(Y_2 + BR^{-1}Y_3)$$

Having X_3 and X_2, use these to find X_1 from the second equation in (7) as:

$$X_1 = R^{-1}[Y_1 + BR^{-1}(Y_2 + BR^{-1}Y_3)]$$

and finally, the first equation in (7) will give X_0 as:

$$X_0 = R^{-1}[Y_0 + BR^{-1}Y_1 + (BR^{-1})^2 Y_2 + (BR^{-1})^3 Y_3]$$

Suppose that we are interested in a t-year planning period with production meeting a constant level of final demand each year. Then,

$$X_0 = R^{-1}[I + BR^{-1} + (BR^{-1})^2 + \dots + (BR^{-1})^t]\,Y \tag{8}$$

Using the power series:

$$I + BR^{-1} + (BR^{-1})^2 + \dots + (BR^{-1})^t = (I - BR^{-1})^{-1}$$

we can rearrange equation (8) as:

$$X_0 = [(I - BR^{-1})R]^{-1}\,Y = (R - B)^{-1}\,Y$$

but $R-B=I-A+B-B=I-A$ and, thus

$$X_0 = (I - A)^{-1}\,Y$$

When t goes to infinite $X_0=X_1=X_2=\dots\dots=X_t=X$. Conclusively, when final demand is constant and the time horizon is infinite, output level is constant and, thus there is no need for capital growth.

Using the terminal condition, equation (5), and assuming that $X_{t+1}=X_t$, we can write equation (5) for t=3 as:

$$\begin{bmatrix} R & -B & 0 & 0 \\ 0 & R & -B & 0 \\ 0 & 0 & R & -B \\ 0 & 0 & 0 & R \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} Y_0 \\ Y_1 \\ Y_2 \\ Y_3 + BX \end{bmatrix} \tag{9}$$

The last equation gives:

$$X_3 = R^{-1}(Y_3 + BX)$$

Working as above, we can compute X_2, X_1 and X_0 as:

$$X_2 = R^{-1}[BR^{-1}(Y_3+BX)+Y_2]$$
$$X_1 = R^{-1}\{BR^{-1}[BR^{-1}(Y_3 + BX) + Y_2] + Y_1\}$$
$$X_0 = R^{-1}[(BR^{-1})^3 BX + (BR^{-1})^3 Y_3 + (BR^{-1})^2 Y_2 + (BR^{-1})Y_1] + Y_0$$

respectively.

The above solution procedure depends on the non-singularity of matrix B. In other words, the inverse of matrix [I–A+B] should exist. The matrix on the left-hand side of equation (5) will be non-singular if and only if R is non-singular.

Alternatively, we can assume that $X_{t+1}=HX_t$, where H is a diagonal matrix, with elements an exogenously determined set of growth rate for each sector. Suppose that the growth rate for period t+1 is known. In this case, equation (5) can be written as:

$$\begin{bmatrix} R & -B & 0 & 0 \\ 0 & R & -B & 0 \\ 0 & 0 & R & -B \\ 0 & 0 & 0 & R-BH \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} Y_0 \\ Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} \qquad (10)$$

The last equation of (10) is:

$$X_3 = (R - BH)^{-1} Y_3$$

In a similar manner as above we can estimate the remaining elements of vector X.

Leontief's method overcomes the singularity problem. It faces, however, a problem regarding consistency with initial conditions. In any case, it is obvious that X(0) is known instead of X(t). Initial conditions are usually treated as given, since nothing can be done to change output level at the initial point of time. Therefore, using Leontief's approach there is no guarantee that the initial conditions will be satisfied. That is, one can project an output level for period 0, which is different than the one actually having at hand.

In the Leontief dynamic model, output increases indefinitely with time, although final demand remains constant. Such an industry needs s period of time, γ, to adjust capital stocks to the rates of output. If the time required for adjustment of the stocks of capital goods to the rate of output is greater than a theoretically determined minimum period, then the dynamic model based on backward difference equations is stable.

Equation (1) can be written as:

$$X_t = AX_t + BX_t + Y_t$$

where X_t denotes differentiation of X with respect to time, or

$$X_t = AX_t + B\left[\frac{(X_t - X_{t-\gamma})}{\gamma}\right] + Y_t \quad (11)$$

Assuming $t = n\gamma$ and considering only the homogeneous part of (11), we have:

$$X_{n\gamma} = AX_{n\gamma} + B\left[\frac{(X_{n\gamma} - X_{(n-1)\gamma})}{\gamma}\right] \quad (12)$$

If $X_{n\gamma} = K\mu^n$ is a non-trivial solution of (12), then μ must satisfy the following equation:

$$\left[(I-A)^{-1}B - \left(\frac{\mu}{\mu-1}\right)I\right]K = 0$$

Matrix $(I-A)^{-1}$ B has a non-negative characteristic root $\theta^* = (\mu^*\gamma)/(\mu^* - 1)$. This characteristic root is considered to be greater than any other. Choose γ large enough so that $-1 < \mu^* < 0$ and $\mu^*/(\mu^* - 1) < 0.5$. Thus,

$$\theta^* / \gamma = \mu^* / (\mu^* - 1) > 0 \text{ or } \theta^* / \gamma < 0.5 \text{ or } \gamma < 2\theta^*$$

If $\gamma < 2\theta^*$, the solution of the homogeneous equation (12) is zero as n goes to infinity and consequently, the solution of equation (11) for the case of constant final demand approaches the static solution. Using difference equations, a better representation of the Leontief dynamic model is obtained because the time period required for adjustment of stocks of capital goods to output cannot be taken as infinitesimal.

Ara (1959) proved that both, before and after aggregation, systems are equivalent with respect to the dynamic stability condition. In other words, the stability condition of a dynamic model is irrelevant to the degree of aggregation of a system.

IV. FORWARD-IN-TIME-INTEGRATION SOLUTION OF THE DYNAMIC SYSTEM

Kendrick's (1972) approach suggests a method which stresses the necessity of integration forward in time using the initial conditions. It overcomes the singularity problem through the use of a partitioning procedure. Assuming that X_0 is initially equal to X_0 and if B^{-1} exists, the dynamic model could be solved by forward integration. The condition that B^{-1} exists, however, rarely holds in multisector models because as explained above, only a small number of sectors produce capital goods.

To overcome this difficulty, matrix B is rearranged. Lets assume that the first n_1 rows ($n_1<n$) of matrix B are non zero and the last $(n-n_1)$ rows contain only zero elements. The partition of R, X and Y can be done in a similar way. Thus, equation (5) can be written as:

$$\left[\begin{array}{c|c} R_{11} & R_{12} \\ \hline R_{21} & R_{22} \end{array}\right]\left[\begin{array}{c} X^{1t} \\ \hline X^{2t} \end{array}\right]-\left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline 0 & 0 \end{array}\right]\left[\begin{array}{c} X^{1t+1} \\ \hline X^{2t+1} \end{array}\right]=\left[\begin{array}{c} Y^{1t} \\ \hline Y^{2t} \end{array}\right] \qquad (13)$$

From this equation it is apparent that the first n_1 rows correspond to sectors which produce capital goods and the rest rows correspond to sectors which produce only goods that are consumed. Equation (13) yields a system of simultaneous difference equations:

$$R_{11}X^{1t} + R_{12}X^{2t} - B_{11}X^{1t+1} - B_{12}X^{2t+1} = Y^{1t} \qquad (14a)$$

$$R_{21}X^{1t} + R_{22}X^{2t} = Y^{2t} \qquad (14b)$$

Solving (14a) for X^{2t} in terms of X^{1t} gives:

$$X^{2t} = R_{22}^{-1}\,(Y^{2t} - R_{21}X^{1t}) \qquad (15)$$

Equation (15) for time t+1 is:

$$X^{2t+1} = R_{22}^{-1}\,(Y^{2t+1} - R_{21}X^{1t+1}) \qquad (16)$$

Substituting (15) and (16) into (14a) yields:

$$X^{1t+1} = (B_{12}R_{22}^{-1}R_{21} - B_{11})^{-1}Y^{1t} - R_{12}R_{22}^{-1}Y^{2t} + B_{12}R_{22}^{-1}Y^{2t+1} - (R_{11} - R_{12}R_{22}^{-1}R_{21})X^{1t} \qquad (17)$$

When X^{10} is equal to X_0^1, equation (17) can be integrated forward to obtain X^{1t} for all t, since final demand is exogenously given. Then, equation (15) can be used to obtain X^{2t} for all t. This approach depends only on the existence of R_{22}^{-1}

and of $(B_{12}R_{22}^{-1}R_{21}-B_{11})^{-1}$. The above conditions are fulfilled because R_{22} and B_{11} are non-singular. The advantage of this approach is that it works by using i-nitial conditions.

Kreijger and Neudecker (1976) compared Leontief's with the forward in time integration approach. To make this comparison feasible, transform equation (13) in the following way:

$$\left[\begin{array}{c|c} R_{11} & R_{12} \\ \hline 0 & 0 \end{array}\right]\left[\begin{array}{c} X^{1t} \\ \hline X^{2t} \end{array}\right]-\left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline R_{21} & R_{22} \end{array}\right]\left[\begin{array}{c} X^{1t+1} \\ \hline X^{2t+1} \end{array}\right]=\left[\begin{array}{c} Y^{1t} \\ \hline Y^{2t} \end{array}\right]$$

or

$$B^{*}X^{t+1} = R^{*}X^{t} - Y^{t} \qquad (18)$$

By taking Penrose's generalized inverse of matrix B as B^{-}, equation (4), the basic equation in Leontief's approach, can be written as:

$$X^{t+1} = B^{-}R^{*}X^{t} - B^{-}Y^{t} + (I - B^{-}B)p \qquad (19)$$

where p is arbitrary and

$$B^{-} = \left[\begin{array}{c|c} B_{11}^{T} & 0 \\ \hline B_{12}^{T} & 0 \end{array}\right]\left[\begin{array}{c|c} \left(B_{11}B_{11}^{T}+B_{12}B_{12}^{T}\right)^{-1} & 0 \\ \hline 0 & 0 \end{array}\right] =$$

$$= \left[\begin{array}{c|c} B_{11}^{T}\left(B_{11}B_{11}^{T}+B_{12}B_{12}^{T}\right)^{-1} & 0 \\ \hline B_{12}^{T}\left(B_{11}B_{11}^{T}+B_{12}B_{12}^{T}\right)^{-1} & 0 \end{array}\right]$$

Using B^{-} instead of B^{-1} reveals that matrix B is singular, which means that it is impossible to calculate its inverse; thus, the generalized inverse should be used. Upon premultiplication by B^{*}, which is non-singular, equation (18) becomes:

$$B^{*}X^{t+1} = B^{*}B^{-}R^{*}X^{t} - B^{*}BY^{t} + B^{*}(I - B^{-}B)p \qquad (20)$$

It should be proved that $B^{*}B^{-}R$ is equal to R^{*} in order to be able to show that equation (19) is equal to equation (20). After the necessary multiplications, it is easy to prove that for the first n_1 sectors, equations (18) and (20) are equivalent. These two approaches can be made identical if we select:

$$B^{*}p = R^{*}X^{t} - \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} Y^{t} + \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} Y^{t+1}$$

Schinnar (1978) developed an alternative approach based on the forward-in-time-integration method. However, he proposed a method which could be used not only when we have the initial conditions. His procedure estimates each year output level based on initial conditions that contain information for more than one years. The advantage of this approach is that uses more historical data than Kendrick's ones. Thus, this method generates more realable estimations of output level.

Schinnar begins from the same point as Kendrick, but for simplicity the basic equation of the dynamic model is written as:

$$X_t - PX_{t+1} = C_t \tag{21}$$

where $P=[I-A+B]^{-1}B$ and $C_t = [I-A+B]^{-1}Y_t$, or in matrix notation:

$$\begin{bmatrix} I & -P & \dots & \dots & \dots & 0 \\ 0 & I & -P & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & I & -P \end{bmatrix} \begin{bmatrix} X^{0} \\ \dots \\ \dots \\ X^{t} \\ X^{t+1} \end{bmatrix} = \begin{bmatrix} C^{0} \\ \dots \\ \dots \\ C_t \end{bmatrix}$$

or

$$\begin{bmatrix} X^{0} \\ \dots \\ \dots \\ X^{t} \end{bmatrix} = \begin{bmatrix} I & P & P^{2} & \dots & \dots & P^{t} \\ 0 & I & P & \dots & \dots & P^{t-1} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & I & P \\ 0 & \dots & \dots & \dots & 0 & I \end{bmatrix} \begin{bmatrix} C_0 \\ \dots \\ \dots \\ C_t \end{bmatrix} + \begin{bmatrix} P^{t+1} \\ P^{t} \\ \dots \\ \dots \\ P \end{bmatrix} X^{t+1} \tag{22}$$

From equation (22), X^0 can be estimated as:

$$X^{0} = C_0 + PC_1 + P^{2}C_2 + \dots + P^{t}C_t + P^{t+1}X_{t+1} \tag{23}$$

Assuming that X^0 denotes an initial output level for $t=0$, equation (23) gives:

$$P^{t+1}X_{t+1} = X^{0} - (C_0 + PC_1 + P^{2}C_2 + \dots + P^{t}C_t)$$

Using the Moore-Penrose generalized inverse, the output level for the period $t+1$ can be estimated as:

$$X_{t+1} = (R^{t+1})^{-} [X^{0} - (C_0 + PC_1 + P^2C_2 + \ldots + P^tC_t)] + [I - (R^{t+1})^{-} R^{t+1}]y \quad (24)$$

where y is arbitrary. Substituting equation (24) into (23), the remaining components of the output vector are obtained.

Alternatively, by assuming that the initial conditions consitute of more than one years' information, for example two years, (23) gives:

$$\begin{bmatrix} P^{t+1} \\ P^{t} \end{bmatrix} X^{t+1} = \begin{bmatrix} X^{0} \\ X^{1} \end{bmatrix} - \begin{bmatrix} I & P & P^2 & \ldots & \ldots & P^t \\ 0 & I & P & \ldots & \ldots & P^{t-1} \end{bmatrix} \begin{bmatrix} C_0 \\ \ldots \\ \ldots \\ C_t \end{bmatrix} \quad (25)$$

Substituting equation (25) into (22), the remaining components of the output vector are obtained.

V. A MODIFIED LEONTIEF DYNAMIC MODEL

Almon (1962) constructed a model to present the dynamic input-output theory. The basic assumptions included are quite different from those of Leontief. The modified assumptions contained in Almon's model are basically five: (a) the flow coefficients are not considered constant but may have linear trends, which are taken as known; (b) consumer demand is a function of population and the real wage rate; (c) all industries have a Cobb-Douglas production functions and thus, substitution of capital and labour is allowed; (d) investment results not only from growth of output but also from the substitution of capital for labour, as the wage goes up and productivity of new capital increases; and (e) the growth path of the wage rate is adjusted so that the resulting final demand gives rise to output, which provides full employment for an exogenous determined labour force. Furthermore, short-term movements will be given scant attention in this model.

Almon developed two variants of this model: one in which constant technological level is assumed and the other in which we can make forecasts of the output of each sector in the presence of technical change. In the former, the analysis is divided in two steps: firstly, the model becomes the Leontief dynamic system if it is known that full employment can be maintain with a forecast of a constant wage rate. Secondly, the model can also be developed for the variable wage rate case.

Almon model uses a Cobb-Douglas production function with constant return to scale:

$$X_j = A_j E_j^{a_j} K_j^{1-a_j} \tag{26}$$

Furthermore, it is assumed that prices of goods and cost of funds will not change over the period, but wages can vary with time even through the wage rate in each sector remains a constant multiple of the wage index. The wage rate varies from industry to industry but through time all move up and down by the same percent. Since material cost are fixed, only labor and capital costs are subject to minimization. Hence, if we let r to be the constant costs of capital, each industry minimizes the following objective function:

$$W(t)\,E(t) + rK(t)$$

subject to (26) or, using a Langrangian formulation:

$$L = W(t)E(t) + rK(t) - \delta(t)[X(t) - AE^{a}(t) - K^{1-a}(t)]$$

From this equation we can derive an equation that explains the investment formulation, namely:

$$\frac{\dot{K}_j}{\dot{K}_i} = \frac{\dot{X}_j}{\dot{X}_i} + a_j\left(\frac{\dot{W}_j}{\dot{W}_i}\right) \tag{27}$$

where the dot denotes the first derivative with respect to time. This equation indicates that the growth of capital stock is a function of output level and of wage rate.

Consumer demand for the output of each industry is a function of the wage rate and of total employed labor force, P(t), which is exogenously given. Thus,

$$C(t) = C_0 W_0 P(t) + C_1 [W(t) - W_0]\, P(t)$$

Government demand and net exports are both exogenously determined and hence final demand is a function of the wage rate and time, as $F = F(w,t)$. By assuming that supply equal demand in each industry:

$$X_i = \sum_{j=1}^{n} a_{ij} X_j + \sum_{j=1}^{n} b_{ij}^{*} K_j + F(W,t) \tag{28}$$

and from equilibrium in labor market:

$$L = \sum_{j=1}^{n} E_j = \sum_{j=1}^{n} \left(\frac{X_j}{A_j K_j^{1-a_j}} \right)^{\frac{1}{a_j}} \tag{29}$$

Moreover, by assuming that we are dealing with the case of constant wage, which means that W(t) = 1 for t > 0 and W(t) = 0, from equation (26) we have that the capital-output ratio remains constant in each industry. Defining b_{ij} as $b_{ij} = (K_j/X_j)\, b^*_{ij}$, which is constant, equation (28) becomes:

$$X_i = \sum_{j=1}^{n} a_{ij} X_j + \sum_{j=1}^{n} b_{ij} K_j + F_i(W,t)$$

or in matrix notation:

$$X = AX + BX + F \tag{30}$$

From a mathematical point of view, (30) is a non-homogeneous system of linear differential equations with constant coefficients. As each differential equation, its general solution is made up of any particular solution plus the general solution of the homogeneous part of equation (30).

Using Neumann expansion, the particular solution of equation (30) is:

$$X = (I-A)^{-1}F + (I-A)^{-1}B(I-A)^{-1}F + \ldots + [(I-A)^{-1}B]^n (I-A)^{-1}F^n \tag{31}$$

where $(I-A)^{-1}F$ represents the total production needed if the economy was static with no growth in final demand and $(I-A)^{-1}B(I-A)^{-1}F$ represents the total direct and indirect requirements for the production of capital which should be produced to make the next years' extra final demand feasible. The higher order terms are necessary if and only if the function of final demand is non-linear.

The general solution of the homogeneous part of (30) is given by:

$$X(t) = C_1 U_1 e^{\varphi_1 t} + \ldots + C_n U_n e^{\varphi_n t} + P(t)$$

where φ are the characteristics roots of $B^{-1}(I-A)$ and the U's are the corresponding characteristics vectors. The standard rule to calculate the constants $C_1, \ldots, C_n$ is to choose them such that to make X(0) equal to some observed initial value of X.

According to Almon, the basic point in this part of the model is that equation (30) should be considered as a system of equations of equilibrium dynamics. The system is satisfied when the economy is growing in equilibrium, but is does not offer any information about how it will behave if it is not in equilibrium and it does not indicate what should be done to reach an equilibrium. Of course, this model does not have any value when dealing with bussiness-cycle models because it provides relations which hold when growth is steady and balanced.

In the case of variable wage, it is assumed that the employment constraint is a non-linear function. Thus, equation (28) becomes:

$$(I-A)X - BX = F(W,t) + B^{*}aK\left(\frac{\dot{W}}{W}\right) \tag{32}$$

Using the above equation and by assuming that all of the second-order terms in the resulting expansion are negligible, we can rearrange equation (31) as:

$$X = (I-A)^{-1}\left\{F(W,t) + B^{*}aK\left(\frac{\dot{W}}{W}\right) + B(I-A)^{-1}F\right\} \tag{33}$$

The solution method of the system of equation (32) consists of two major parts. In the first part, calculate K and w for gives value of K and w. With given K and w, we can make a guess w, calculate X by equation (33) and find how much employment this X gives by equation (29). In the second part, calculate K(t) and w(t) for t = h, 2h, 3h, etc. The value at t=0 is assumed to be known. Using the second -order Runge-Kutta method, set K and w equal to their initial values and make a guess for w. After computing X by equation (33), the labor required for this X can be found by equation (29). If E is high enough, we can continue the calculations. If it is not, we should return and make a new guess for w. Using these results as initial conditions we can repeat the process to forecast the second period.

In contrast, when there is technological change in the economy, the matrix A is defined as a linear function of time: A(t)=A1+A2t, where A1 and A2 are constant and known matrices. The final demand and production functions are same as above:

$$F = F(W,t) = F1+F2+\{[W(t)-1](F3+F4t)\}$$
$$X_j = A_j E_j^{a_j} K_j^{1-a_j}$$

respectively. If u is the rate of increase of productivity and d is the deprecation rate, then from a series, I(t), of gross investment in constant money units, the function for K(t) can be derived:

$$K_j(t) = \int_{-\infty}^{t} e^{U\gamma d(t-\gamma)}\, I(\gamma)\, d\gamma$$
$$K = e^{ut} I(t) - dK$$

Each industry minimize the function $w(t)E(t)+^{pe-ut}K(t)$ subject to the constraint (26), where funds denoted as p. Using a Lagrangian formulation with control variables E and K, the function of capital level in each industry is found as:

$$K_j(t) = C_j X_j(t) W(t)^{a_j} e^{a_j u_j t} \tag{34}$$

where C_j is the desired capital-output ratio at t=0. Differentiating (30) with respect to time gives:

$$I(K) = C W^{a_j} e^{(a_j-1)u_j t} \left[X + \left(a \left\{ \frac{\dot{W}}{W} \right\} \right) + (au+d) \right]$$

and total investment demand is:

$$Inv(t) = BG1(t, W) + BG2(t, W)DX(t)$$

where D is the differential operator.

The complete system of balance equations for the economy are:

$$X(t) = A(t)X(t) + Inv(t) + F(t, W) \tag{35}$$

$$E(t) = L(t) = \sum_{j=1}^{n} C_j X_j W^{a_j-1} e^{(a_j-1)u_j t} \tag{36}$$

The above system of equations comprise n+1 equations with n+1 unknown functions, $X^1(t),...,X^n(t)$ and w(t). To solve the above system firstly it should be assumed a particular function for $w(t)=W(t,p)=1+p_1t+p_2t^2+p_3t^3$ where p is a vector of parameters in the wage function. Making an intitial guess for p as p(0), with w(t,p(0)) solve equation (35) for X(t) and use equation (36) to calculate E(t). Comparing the founded E(t) with L(t) make a second guess for p and redo the same procedure until E(t)=L(t).

VI. CONCLUDING REMARKS

The dynamic input-output model is much more helpful than a static one in analyzing impacts in an economy because reality is reflected more accurately. But problems arise with respect to the selection of data that explain the interindustry flows of capital goods and their comparison with actual ones. Even if we overcome the singularity problem of matrix B, using the generalized inverse, another problem arises. How can we find available data to construct this matrix? The only solution is to make a survey for a small number of sectors and approximate the resutls for all sectors. This procedure needs more time and probably when we prepare the technical coefficients some technological change will take place and thus, these coefficient become inconsistent.

The assumption used in the dynamic model are more realistic but still quite restrictive. Perhaps they are responsible for the inconsistency of forecasting. The difference that arise when we compare the observed and forecasting levels of output dictate from the assumption that substitution between factors of production is not allowed and therefore, the technological level remains constant. This assumption is too restrictive because each economy exhibits technological change over time, whatever growth rate was provided. Furthermore, it is impossible to consider some sectors, as electric power, oil, manufacturing, communication and so fourth, without technological change. In this case Almon model is helpful since it correlates wage rate, technological innovations and output level with the exogenously determined components of final demand over time. Difficulties with this model arises, however, with the selection of data and their availability.

The above discussion of the alternative solutions of the dynamic input-output model indicates that the use of the forward-in-time-integration approach is necessary. The advantage of this approach is that it is based on initial (actual) conditions. Thus, by knowing the actual level of final demand for the initial period of time (first year) and by assuming a growth rate, the output for the following years can be forecasted. Of course, Schinnar approach, which is based on initial conditions for more than one years, entails forecasts that are correlated with actual observations of the past and probably it yields more efficient projections.

The main problem with the backward-in-time-integration approach is that it does not prevent the determination of initial (actual) years' output levels. Using this approach, it is probable that projections will be inconsistent with the initial conditions because we select an arbitrary value of terminal output.

The dynamic input-output model is not widely used in applied research mainly due to data collection problems. They are very few studies that made use of this model. Miernyk (1970) used it to examine alternative economic development strategies for the state of West Virginia; Richardson (1972) developed a dynamic input-output model at a regional level for U.S.A.; and Miernyk and Sears (1974) used a dynamic specification of input-output to analyze the impacts of pollution-control technologies on regional economies. Unfortunately, the use of the dynamic input-output model stop in the middle of '70s with the development of investment models, the use of rational expectations and the general equilibrium models.

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