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THE REACHABILITY OF A STABLE EDGEWORTH EQUILIBRIUM

PLAN

ABSTRACT

This tract studies the reachability properties of a stable general equilibrium resulting from an Edgeworth trading process. It is shown on the basis of a relaxed hereditary nonlinear differential system that under an appropriate definition of stability and reachability, the reachability of an Edgeworth equilibrium at reasonable time requires one more assumption than those needed for stability. This additional assumption imposes a restriction on the possible time paths of an aggregate utility function.

1. INTRODUCTION*

The issue of how fast a general competitive equilibrium is restored, or formally, the issue of the reachability of such an equilibrium, after a disturbance of it, is an important one: If restoration is slow and disequilibrium persists, there is plenty of room for policy intervention even under perfectly competitive conditions. The issue in hand has not escaped the attention of the literature, which however has been unable so far to offer some answers. This in conjunction with the real–world experience of prolonged disequilibrium situations, has led to the conclusion that there is nontatonnement trading that has to do with market imperfections and/or imperfect competition (see e.g. Nishimura (1922)), and in general, with «fixprice» elements (see e.g. Silvestre (1986)). It is clear that this fixprice microeconomics tries to offer a microfoundation of macroeconomics, a foundation in which general equilibrium is only one out of many other possible outcomes of trading.

Another type of fixprice microeconomics that remains within the realms of «pure» microeconomics (and so call it «orthodox»), argues that the tatonnement process involves adjustments in quantity constraints rather than prices: An auctioneer repeatedly posits constraints and solicits hypothetical offers until an equilibrium is reached. Thus, the nontatonnement, the disequilibrium, model alters the character of the Walrasian tatonnement, but retains the process (see e.g. Samuelson (1986)). The point is that trading in this fixprice approach, takes place only in equilibrium. According to the other strand in fixprice microanalysis, the orthodox approach can not get rid of this Walrasian feature, because it does not define the institutional framework of trading, and trade at non–Walrasian prices requires such a definition (see e.g. Silvestre (1986)). More precisely, both the Walrasian (or Arrow–Debreu) model and the orthodox fixprice microeconomics can not give a unique answer to the question «how many markets are there (in their models)». That is, strict determination of the number of markets enables analytically disequilibrium trading that only accidentally may produce a general equilibrium. At the other end, loose determination of the number of markets admits analytically trade only under a general equilibrium state of affairs.

This paper attempts to resolve this trade–off by taking analysis from the finite dimensional Euclidean space to the (countably) infinite dimensional Ba-

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nach space. No one can argue in the latter space that the number of markets is «perhaps n (one for each good, with the interpretation that in market j good j is exchanged against an unlisted (n+1)th good that disappears after the execution of the exchanges), or possibly (n–1) (with the interpetation that one of the goods, say good n, is a means of payment exchanged against good j in market j), or maybe n(n–1)/2 markets, one for each pair of goods, or even zero markets (with the interpretation that the equilibrium allocation is implemented by means other than trading» (Silvestre (1986, p. 197)). We thus manage to achieve a synthesis in fixprice microeconomics. A synthesis in which disequilibrium exchange is permitted and leads to a general equilibrium. The analysis in Banach spaces allows such exchange, because it also allows the incorporation of the time needed by the «memory of the agents concerning the equilibrium situation» to bring on the full effects of this memory after a disturbance of the equilibrium. That is, the key in our approach is the reachability of the equilibrium, i.e. a matter that could not be tackled in the past because of the above economic theory considerations and because the speed of adjustment of the excess demand in the Euclidean space is parametric. In this manner, the orthodox fixprice microeconomics and the Walrasian paradigm turn out to be special cases in a more general apparatus.

Note that our analysis here does not postulate any a priori reasons, like fixprice elements, for a would-be delay in equilibrium restoration. The delay is modelled as a time parameter that may take on any nonnegative value for any reason. Anyway, I think it is time to proceed to the rigorous elaboration of the topic under study. In what follows, the next section spells out the problem in hand formally in both Euclidean and Banach spaces. Section 3 derives the conditions under which stability and reachability are expected to prevail. Section 4 concludes this paper with further remarks on the nature of our approach and theses.

2. A JOINT STABILITY–AND–REACHABILITY APPROACH

The nontatonnement trading process of Edgeworth leads to a stability theorem via the Liapunov function

$$V(p,W) = -U(W(t)) \tag{1}$$

where p is a price vector, U is a sum of individual utility functions, W is an allocation of endowments, and t is time (see e.g. Varian (1978, p. 192)). In general, if the minimum of the Liapunov function V at time t_0 corresponds to $U_{t_0} = Y_0$, the eq. (1) must satisfy the following conditions:

$$\dot{U}(t) = g(U_t, W(t), t) > 0 \qquad (t \in T: = [t_0, t_1]) \tag{2}$$

$$U_{t_0} = Y_0 \tag{3}$$

and $W(t) \in A(t) \qquad (t \in T)$ (4)

where (i) the rule by which the Cartesian product $K^n[-r,0] \times R^m \times T$ is mapped into the set R^n:g, (ii) the length of the memory of the differential system (2)–(4):r, $0 \leq r < \infty$, $t_1 - r > t_0$, $Y_0 \in K^n[-r,0]$, and (iii) the endowment control space $A(t) \subset R^m$, are fixed, and (iv) in addition,

$$U_t(s): = U(t+s) \qquad s \in [-r,0]$$

Note that the inequality $t_1 > t_0 + r$ reflects the fact that the stronger memory is the less r is and hence, the less the difference $(t_1 - t_0)$.

In this paper, we study the problem of the reachability of Y_0 at time t_1 by controlling W(t) when the equilibrium resulting from an Edgeworth trading process, Y_0, is disturbed at time t_0, (while the trading process may or may not involve disequilibrium exchange). To solve this problem, note that the differential system (2) – (4) is a nonlinear hereditary system, and we are essentially interested in its reachability properties. These properties depend on the choice of the state space C, since the state of the system is given by the function segment U_t. Following Kurcyusz and Olbrot (1977), the choice of C considered here is the Sobolev space $C = D^{n,\infty}[-r,0]$. Also, the reachability properties of the system depend on whether the dimension m of the endowment control space is greater than or less than the dimension n of the Cartesian product $K^n[-r,0] \times R^m \times T$. To determine this, note that as Shubik (1975, p. 557) points out, «in a disequilibrium state every participant must be able independently to choose an allocation». Therefore, we assume that $n \geq m$ and following Colonius (1982) and Warga (1972), we replace W by relaxed, measure–valued, control functions $M \in E$ (5). Substituting W in eq. (2) by M we obtain the following relaxed hereditary differential system:

$$\dot{U}(t) = g(U_t, M(t), t) \qquad (t \in T: = [t_0, t_1]) \tag{6}$$

$$U_{t_0} = Y_0 \tag{3}$$

$$M(t) \in E(t) \tag{5}$$

In this manner, our problem is formally equivalent to finding out the reachability properties of the above relaxed system when the state space is the Sobolev space $C = D^{n,q}[-r,0]$, $\quad 1 \leq q \leq \infty$.

The reformulation of the system (2) – (4) in terms of (3), (5), and (6) enables one to tackle not only the issue of stability, but also the matter of reachability on the basis of the mathematics of functional analysis. In what follows, we assume that a final state Y_0 can be reached via a trajectory U^0 corresponding to a relaxed control $M^0 \in E$. Next, reaching $U(-r)$ at time t_1-r, tracing the velocity function $U(t-t_1)$ on $[t_1-r,t_1]$, and assuming that the neighborhood $Y_0(-r) \in R^n$ can be reached with arbitrarily small deviations from U^0, the hereditary impact on the velocity can be kept small, and we can reach U in a neighborhood of Y_0 in $D^{n,\infty}[-r,0]$ by compensating small deviations from $Y_0 = U^0_{t_1}$.

3. THE CONDITIONS FOR REACHABILITY

Notation, Definitions, Assumptions

Let $K^n(B)$ be the Banach space of continuous functions on the compact set $B \subset R^m$ with values in R^n. Also, let $D^{n,q}[h,l]$, $1 \le q \le \infty$, be the Sobolev space of absolutely continuous function $U:[h,l] \to R^n$. $\dot{U} \in F^n_q[h,l]$ is a q–integrable bounded derivative. The symbol $|\cdot|$ denotes the Euclidean norm in a space of finite dimensions. The norm in the (also linear) Banach space $D^{n,q}[h,l]$ is $\|U\| := \left|\left(|U(h)|, \|\dot{U}\|_{F_q}\right)\right|$. (Both norms obey Pythagoras' theorem, but contrary to the Euclidean norm, the summation in the distance function of the Banach norm goes to infinity.) $R^n \times F^n_q[h,l]$ is a canonical way identifying $D^{n,q}[h,l]$. co B is the convex hull of the interior of a set B in a Banach space. $A(t) \subset A_0(t \in T)$, where $A_0 \subset R^m$ is compact, $t \to A(t)$ is measurable, and $A(t)$ is closed for all $t \in T$. E is the set of M's on $T:=[t_0,t_1]$ with values in the set of Radon probability measures on A_0 having support contained in $A(t)$. Each endowment control W is identified with the relaxed control $e_{W(.)} \in E$, where $e_{W(t)}$ is the point measure concentrated at $W(t) \in A(t)$. T_1 is the final interval $[t_1-r,t_1]$. Finally, N represents the relaxed hereditary differential system.

With this notation in mind, let us make now a couple of definitions.

Definition 1: $g(U_t,M(t),t) := \int_{A_0} g(U_t,w,t)\, M(t)\, dt$

This only says that responsible for the intertemporal course of U are exclusively the consequences of the controls following t_0.

Definition 2: If for each neighborhood L of U^0 in $K^n[t_0-r,t_1]$ there is a neighborhood Q of Y_0 in $D^{n,\infty}[-r,0]$ such that for each $Y \in Q$ there is a trajectory $U \in L$ of N with $Y=U_t$, then $Y_0 \in D^{n,\infty}[-r,0]$ is said to be reachable at time t_1 with a trajectory U^0 of N.

It is self-explanatory that if the initial position Y_0 is to be reachable, it must be because there is one at least trajectory that can restore it.

Finally, assume that,

Assumption 1: $t \to k(M(t)) := \int_{A_0} k(w)\, M(t)\, dw$ is measurable for each $k \in K(A_0)$.

Time is assumed to be measurable in terms of the reallocations of the endowments that are brought about by the controls. That is, time constitutes here a ratio scale (like a weight measure) rather than an interval scale (like temperature measures) as is conventionally assumed. This assumption is important because it allows explicitly disequilibrium trading, which however may or may not take place depending on whether r is zero.

Assumption 2: $g: K^n[-r,0] \times R^m \times T \to R^n$ is continuous in $(Y,w) \in K^n[-r,0] \times R^m$ and measurable in $t \in T$.

The analogue of this assumption in the Euclidean space is that the utility function is at least twice continuously differentiable. This assumption postulates also that utility is measurable at any point in time. It is the neoclassical concept of utility as a strength–of–preferences indicator except that no ratio (measurement) properties are presumed, thus providing ordinal ranking (see e.g. Frisch (1964) and Alt (1971)).

Assumption 3: $q: R_+ \times T \to R_+$ is such that for all $U \in K^n[t_0-r, t_1]$ and $w \in A_0$,

$$|g(U_t, w, t)| \le q(\|U_t\|_\infty, t), \qquad (t \in T).$$

According to this assumption, if the number of markets is infinite, equilibrium restoration (in the final interval time T_1) should be slower vis a vis the finite case. Therefore, in Banach spaces, the instantaneous adjustment towards the equilibrium is precluded even if adjustment is instantaneous in Euclidean spaces.

Assumption 4: g is continuously Frechet differentiable in the first argument, the corresponding derivative $\Delta_1 g(Y,w,t)$ is continuous in (Y,w,t) and for all $w \in A_0$, $\left\|\Delta_1 g(Y,w,t)\right\| \le q(\|Y\|_\infty, t)$, $(t \in T)$, where q is as in assumption 3.

This assumption is rather technical, since it follows from assumptions 2 and 3.

Assumption 5: For a measurable $S \subset T$, for $U^0 \in K^n[t_0-r, t_1]$, and $c \in F^n_\infty(S)$, there is a neighborhood L of $O \in R^n$ such that $L \subset c(t) + \text{co}\, g(U^0, A(t), t)$, $(t \in S)$.

This assumption states that any deviations from U^0 in T_1 are small, and remain about it.

Assumption 6: The trajectory U^0 of N reaches $Y_0 \in D^{n,\infty}[-r,0]$ at time t_1, and there are $u>0$ and a neighborhood L of $O \in R^n$ such that

$$L \subset -\dot{U}^0(t) + \text{co}\, g\left(U_t^0, A(t), t\right), \qquad (t \in [t_1-r-u, t_1-r]).$$

Finally, it is assumed that Y_0 is reached precisely at $t=t_1$ and that any deviations from U^0 during t_1-r are quickly corrected. Note that this assumption imposes stability beforehand and can not be used to prove stability. It also places a time constraint not only on the reachability of Y_0 but also on the deviations from U^0 before the exhaustion of the memory.

Theoretical Propositions

The first result referes to the stability of the hereditary differential system.

Proposition 1: If assumptions 1–5 are true, then there are $e>0$ and a neighborhood L' of $O \in R^n$ such that for all U with $\|U-U^0\|_\infty < e$,

$$L' \subset c(t) + \text{co}\; g(U_t, A(t), t) \qquad (t \in S).$$

Proof: The norm $\|U-U^0\|$ is simply the infinite dimensional version of the (square root of the) quadrartic–loss–function form of the Liapunov function. In Euclidean spaces, simple differentiation of this function would suffice to prove stability. In Banach spaces, however, one has to assume additionaly that deviations for U^0 in T_1 are small, (assumption 5), because recall that in T_1 there is no memory. Consequently, if deviations are sizeable, «quadratic–loss–function behavior» may indeed return the system back to U^0, but to a point close to Y_0, not to Y_0. This is actually a sketch of the proof. The formal proof follows Colonius (1982) and Warga (1972), and is outlined in the Appendix.

Proposition 2: If assumption 6 and proposition 1 are true, then $Y_0(-r)$ is reachable with U^0 at time t_1-r and Y_0 is reachable with U^0 at time t_1.

Proof: If would–be deviations from U^0 in t_1-r are small, or more precisely, quickly corrected by quadratic–loss–function behavior, then $Y_0(-r)$ is reachable exactly at the end of the interval t_1-r along the lines of the previous proof. Note, however, that previously we had to have assumption 5 because we were in T_1. Now that memory is present, the interaction between it and quadratic–loss–function behavior suffice to keep any deviations from U^0 small and prove the first part of proposition 6, namely that the system moves essentially along U^0 during t_1-r. The whole proposition states that such a movement is a prerequisite in order to reach Y_0 at $t=t_1$: Arriving at $Y_0(-r)$ as fast as possible is a prerequisite for the deviation–controlling behavior in T_1 to get us at $t=t_1$ back not to any point on U^0 but to Y_0 exactly by assumption 6. Note that proposition 1 establishes stability but it does not say when: before t_1 or at t_1? (Certainly not after t_1 because U^0 has by definition a time horizon for up to t_1.) It is for this comparison and for technical reasons (that render the proof of proposition 2 easier), that we need assumption 5 and proposition 1 behind proposition 2; otherwise, assumption 6 would suffice for our purposes here. Again, these considerations offer a sketch of the proof. The formal proof follows the mathematics of Colonius (1982) and Warga (1972), and is summarized in the Appendix.

4. CONCLUDING REMARKS

One might argue that our reachability–and–stability result is of limited empirical importance, since it follows directly from assumption 6 given that the system behaves in a quadratic–loss–function fashion: Our result is simply not falsifiable in Popper's sence; it is always true and hence, always false. Indeed, I admit that this assumption is quite restrictive. I used it because that is as far as my mathematics can get me. Yet, this does not impair the usefulness of the approach followed in general. A better mathematical economist than I would only need to modify assumption 6 to get falsifiable and hence, empirically meaningful statements. But, that is the point of my work: Once its conclusions have become probabilistic, they will have formed a general theory of fixprice microeconomics and a sound microfoundation of macroeconomics. It is from this point of view that I think the work in hand deserves some attention.

APPENDIX

Proof of Proposition 1: Consider L an n–simplex with vertices $x_0, x_1, \ldots, x_n$. Then there are $M_i \in E$, $i=0, 1, \ldots, n$, with

$$x_i = c(t) + g\left(U_t^0, M_i(t), t\right) \qquad (t \in S)$$

Also, $g(U_t, M_i(t), t)$ is uniformly close to $g\left(U_t^0, M_i(t), t\right)$ for small $\|U-U^0\|_\infty$. In this manner, there are $e>0$ such that for all U with $\|U-U^0\|_\infty<e$ and for all $t \in S$ the points $g\left(U_t, M_i(t), t\right)$ are equally well vertices of n–simplices containing a neighborhood L′ of $O \in R^n$. The proof becomes complete when one notes that convex combinations of $g\left(U_t, M_i(t), t\right)$ remain in co $g\left(U_t, A(t), t\right)$.

Proof of Proposition 2: According to proposition 1, a set H can be defined so as for all $e>0$

$$H := \{b \in R^n : b = U(t_1-r) \text{ for some } U \in K^n[t_0-r, t_1] \text{ with}$$
$$U(t) = U^0(t) \text{ for } t \in [t_0-r, t_1-r-u],\ \|U-U^0\|_\infty < e$$
$$\text{and } \dot{U}\big|[t_1-r-u, t_1-r] \in P, \text{ P being a neighborhood}$$
$$\text{of } \dot{U}^0\big|[t_1-r-u, t_1-r] \in F_\infty^n \text{ such that for all U}$$
$$\text{with } \|U-U^0\|_\infty < e \text{ and all } c \in P$$
$$c(t) \in \text{co } g(U_t, A(t), t)\ \left(t \in [t_1-r-u, t_1-r]\right)\}$$

forms a neighborhood of $Y_0(-r)$ in R^n. Also, according to Warga (1972), U is a trajectory of N reaching b by definition. In this manner, U^0 reaches $Y_0(-r)$ at time t_1-r. The reachability of Y_0 at time t_1 follows immediately by noting that the set

$$I := \left\{Y \in D^{n,\infty}[-r,0] : Y = U_{t_1} \text{ for some } U \in K^n[t_0-r, t_1] \text{ with}\right.$$
$$U(t) = U^0(t) \text{ for } t \in [t_0-r, t_1-r-u],\ \|U-U^0\|_\infty < e,$$
$$\text{and } \dot{U}\big|[t_1-r-u, t_1] \in P, \text{ P being a neighborhood of}$$
$$\left.\dot{U}^0\big|[t_1-r-u, t_1] \in F_\infty^n\right\}$$

forms a neighborhood of Y_0 in $D^{n,\infty}[-r,0]$.

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