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STRUCTURAL APPROACH TO SLIPPAGE PROBLEMS FOR EXPONENTIAL SAMPLES

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ABSTRACT

The structural method of inference is used to identify outlying sub-samples in a set of samples with exponential error components. Models for location shift, scale change, and combination of both are examined. The structural approach leads to the distributions of the slippage parameters for these models. Therefore, inference statements such as interval estimates and hypothesis testing can be established.

1. INTRODUCTION

Slippage problems have been considered in the statistical literature by employing non-parametric, classical parametric, and Bayesian techniques; see Barnett and Lewis (1984) and references therein. Slippage problems can be described as follows: Suppose we have a data set which can be divided into k distinct sub-samples according to some characteristics. Alternatively, we may have k random samples of sizes $n_1, n_2, \dots, n_k$ coming from a common family of distributions, where at least one $n_r > 1$, $r = 1, 2, \dots, k$. The objective is to test whether one or more of the samples have undergone slippage relative to the others, by having a different location parameter, or a different scale parameter, or both.

Slippage problems can be distinguished according to two classification criteria:

a. Labelled - Unlabelled: If it is known in advance which samples are candidates for slippage, then we have a labelled slippage problem; otherwise we have an unlabelled slippage problem. Most of the work found in the literature concerning slippage tests deals with the unlabelled case. However, in the majority of cases there exists in advance some explicit or implicit information regarding the index of the potential samples, which are suspected to have undergone slippage. Thus, the usually unlabelled problems often reduce to labelled ones.

b. Specified - Unspecified: If we specifically test for downward or upward slippage with respect to location, and shrinkage or inflation with respect to scale, then we have a specified slippage problem. Most of the existing approaches must use different test statistics depending on the direction of slippage. This restricts these methods to one-sided alternative hypotheses.

In this paper we consider labelled and unspecified slippage problems for exponential samples.

2. DESCRIPTION OF THE PROBLEMS

Initially we combine all observations from the k sub-samples. Then, let $y_1, y_2, \dots, y_n$ be the combined observations from the non-slipped majority of the samples. Also, let $z_1, z_2, \dots, z_m$ be the combined observations from the supposedly slipped samples. Using this notation, the three slippage models can now be given:

Location Shift: This model can be described with the equations

$$y_i = \mu + \sigma e_i \quad , \quad i = 1, 2, \dots, n$$
$$z_j = \mu + \lambda + \sigma \varepsilon_j \quad , \quad j = 1, 2, \dots, m$$

Scale Change: In this model, it is assumed without loss of generality that the observations have a location parameter equal to zero. Then, the model can be formulated as

$$y_i = \sigma e_i \quad , \quad i = 1, 2, \dots, n$$
$$z_j = \gamma \sigma \varepsilon_j \quad , \quad j = 1, 2, \dots, m$$

Location Shift and Scale Change: Combining the two previous models, we can also consider the model

$$y_i = \mu + \sigma e_i \quad , \quad i = 1, 2, \dots, n$$
$$z_j = \mu + \lambda + \gamma \sigma \varepsilon_j \quad , \quad j = 1, 2, \dots, m$$

In the above models, the parameters μ, $\mu \in R$, and σ, $\sigma \in R^+$, are the location and scale parameters of the majority of the observations y_i. Also, the parameters λ, $\lambda \in R$, and γ, $\gamma \in R^+$, are the slippage location and scale parameters of the possibly slipped observations z_j. The random error components e_i's and ε_j's are assumed to be independent and identically distributed according to the standard exponential distribution Exp(1). That is,

$$f(e_i)\, de_i = \exp(-e_i)\, de_i, \qquad e_i \in R^+, \qquad \text{and}$$

$$f(\varepsilon_j)\, d\varepsilon_j = \exp(-\varepsilon_j)\, d\varepsilon_j, \qquad \varepsilon_j \in R^+.$$

The structural method of inference (Fraser 1968, chapter 2, pages 49-74) is employed to derive the distributions of the slippage parameters. Therefore, interval estimates for λ and γ can be obtained. Tests for the hypotheses $\lambda = 0$ and $\gamma = 1$ can be based on these interval estimates.

3. LOCATION SHIFT MODEL

The location shift model can be expressed in structural form as

$$\mathbf{X} = \boldsymbol{\Theta} \bullet \mathbf{E}$$

where **X** is a 3x(n+m) response matrix, $\boldsymbol{\Theta}$ is a 3×3 transformation matrix, and **E** is a $3\times(n+m)$ error matrix, such that

$$X = \begin{bmatrix} 1' & 1' \\ 0' & 1' \\ y' & z' \end{bmatrix}, \qquad \Theta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \mu & \lambda & \sigma \end{bmatrix}, \qquad \text{and} \qquad E = \begin{bmatrix} 1' & 1' \\ 0' & 1' \\ e' & \varepsilon' \end{bmatrix}$$

with $\quad y' = [y_1 \;\; y_2 \ldots y_n], \qquad z' = [z_1 \;\; z_2 \ldots z_m],$

and $\quad e' = [e_1 \;\; e_2 \ldots e_n], \qquad \varepsilon' = [\varepsilon_1 \;\; \varepsilon_2 \ldots \varepsilon_\mu].$

Direct structural analysis yields the structural distribution of the parameters given the data (Armenakis 1988, pages 67-78) as

$$g^*(\mu, \lambda, \sigma / y, z)\, d\mu\, d\lambda\, d\sigma = \frac{nm}{\Gamma(n+m-2)} \; \cdot$$

$$\cdot \; \exp\left\{-\frac{1}{\sigma}\left[n(y_{(1)} - \mu) + m(z_{(1)} - \lambda - \mu) + S_x\right]\right\} \; \frac{S_x^{n+m-2}}{\sigma^{n+m+1}} \; d\mu\, d\lambda\, d\sigma\,, \tag{3.1}$$

where $y_{(1)}$ and $z_{(1)}$ are the first order statistics of the observations y_i and z_j respectively, and

$$S_x = \sum_{i=1}^{n} y_i - n y_{(1)} \; + \; \sum_{j=1}^{m} z_j - n z_{(1)}$$

Also, $\mu < \mu_0 = \min\{y_{(1)}, z_{(1)} - \lambda\}$, $\lambda \in R$, and $\sigma \in R^+$. To find the distribution of the slippage location parameter λ, we need to integrate expression (3.1) with respect to σ and μ. Integrating over σ we obtain that

$$g^*(\mu, \lambda / y, z)\, d\mu\, d\lambda = n\, m\, (n+m-1)\, (n+m-2) \;\; S_x^{n+m-2} \;\; \cdot$$

$$\cdot \;\; \left[n(y_{(1)} - \mu) + m(z_{(1)} - \lambda - \mu) + S_x\right]^{-(n+m)} d\mu\, d\lambda \tag{3.2}$$

To integrate expression (3.2) over μ, $\mu<\mu_0$, it is necessary to distinguish between the cases $\mu_0 = y_{(1)}$ and $\mu_0 = z_{(1)} - \lambda$.

Case 1: Let $\mu_0 = y_{(1)}$, which also implies that $y_{(1)} < z_{(1)} - \lambda$ and therefore $\lambda < z_{(1)} - y_{(1)}$. Then, the structural distribution for the slippage location parameter λ is given by

$$g_1^*(\lambda/\mathbf{y},\mathbf{z})\,d\lambda = n\,m(n+m)^{-1}(n+m-2)\ S_x^{n+m-2}\ \cdot$$
$$\cdot\ \left[-m\lambda+m(z_{(1)}-y_{(1)})+S_x\right]^{-(n+m-1)}\ d\lambda \tag{3.3}$$

Case 2: In this case let $\mu_0 = z_{(1)}-\lambda$, which also implies that $z_{(1)}-\lambda < y_{(1)}$ and therefore $z_{(1)} - y_{(1)} < \lambda$. Then, the structural distribution for the slippage location parameter λ is obtained as

$$g_2^*(\lambda/\mathbf{y},\mathbf{z})\,d\lambda = n\,m\,(n+m)^{-1}(n+m-2)\ S_x^{n+m-2}\ \cdot$$
$$\cdot\ \left[n\lambda-n(z_{(1)}-y_{(1)})+S_x\right]^{-(n+m-1)}\ d\lambda \tag{3.4}$$

Using expressions (3.3) and (3.4), it can be shown that

$$\int_{-\infty}^{+\infty} g^*(\lambda/\mathbf{y},\mathbf{z})\ d\lambda = \int_{-\infty}^{z_{(1)}-y_{(1)}} g_1^* = (\lambda/\mathbf{y},\mathbf{z})\ d\lambda + \int_{z_{(1)}-y_{(1)}}^{+\infty} g_2^*(\lambda/\mathbf{y},\mathbf{z})\ d\lambda =$$

$$= \frac{n}{n+m}+\frac{m}{n+m} = 1$$

which verifies the validity of the structural distribution for the parameter λ.

The structural distribution of λ can now be used to construct $100(1\text{-}\alpha)\%$ interval estimates (λ_L, λ_U) for the parameter λ.

Let $\frac{\alpha}{2}<\frac{n}{n+m}$ and $\frac{\alpha}{2}<\frac{m}{n+m}$. Then the lower limit λ_L can be obtained from the integration of expression (3.3) as

$$\int_{-\infty}^{\lambda_L} g_1^*(\lambda/\mathbf{y},\mathbf{z})\ d\lambda = \frac{\alpha}{2},$$

which gives

$$\lambda_L = z_{(1)}-y_{(1)} + \frac{S_x}{m} - \frac{S_x}{m}\left[\frac{2n}{(n+m)\alpha}\right]^{\frac{1}{n+m-2}} \tag{3.5}$$

Similarly, the upper limit λ_U can be derived from the integration of expression (3.4) as

$$\int_{\lambda_u}^{+\infty} g_2^* (\lambda / \mathbf{y},\mathbf{z}) \; d\lambda = \frac{\alpha}{2},$$

which yields

$$\lambda_U = z_{(1)} - y_{(1)} + \frac{S_x}{n} + \frac{S_x}{n}\left[\frac{2m}{(n+m)\alpha}\right]^{\frac{1}{n+m-2}}$$

If $\frac{\alpha}{2} < \frac{n}{n+m}$ and $\frac{m}{n+m} < \frac{\alpha}{2}$, then the lower limit λ_L is again given by equation (3.5). Also, in this case the upper limit λ_U can be obtained from the integration of expression (3.3) as

$$\int_{-\infty}^{\lambda_U} g_1^* (\lambda / \mathbf{y},\mathbf{z}) \; d\lambda = 1 - \frac{\alpha}{2},$$

which gives

$$\lambda_U = z_{(1)} - y_{(1)} + \frac{S_x}{m} - \frac{S_x}{m}\left[\frac{2n}{(n+m)(2-\alpha)}\right]^{\frac{1}{n+m-2}}$$

4. SCALE CHANGE MODEL

In the scale change model, it is assumed without loss of generality that the location parameter of the data is zero. Then, the model is expressed in structural form as

$$\mathbf{X} = \boldsymbol{\Theta} \bullet \mathbf{E}$$

where $\mathbf{X}$ is a 2x(n+m) response matrix, $\boldsymbol{\Theta}$ is a 2×2 transformation matrix, and $\mathbf{E}$ is a 2x(n+m) error matrix, such that

$$\mathbf{X} = \begin{bmatrix} \mathbf{y}' & \mathbf{0}' \\ \mathbf{0}' & \mathbf{z}' \end{bmatrix}, \qquad \boldsymbol{\Theta} = \begin{bmatrix} \sigma & 0 \\ 0 & \gamma\sigma \end{bmatrix}, \text{ and } \mathbf{E} = \begin{bmatrix} \mathbf{e}' & \mathbf{0}' \\ \mathbf{0}' & \boldsymbol{\varepsilon}' \end{bmatrix}$$

with $\mathbf{y}' = [y_1\ y_2 \ldots y_n]$, $\mathbf{z}' = [z_1\ z_2 \ldots z_m]$

and $\mathbf{e}' = [e_1\ e_2 \ldots e_n]$, $\varepsilon' = [\varepsilon_1\ \varepsilon_2 \ldots \varepsilon_m]$.

Applying the techniques of structural inference, we can obtain the structural distribution of the parameters (Armenakis 1988, pages 79-86) as

$$
\begin{aligned}
g^*(\sigma, \gamma / \mathbf{y}, \mathbf{z})\, d\sigma\, d\gamma &= \frac{1}{\Gamma(n)\ \Gamma(m)} \\
&\cdot \exp\left[-\frac{S_x}{\sigma}\left(1+\frac{C_x}{\gamma}\right)\right] \frac{C_x^m}{\gamma^{m+1}} \frac{S_x^{n+m}}{\sigma^{n+m+1}}\, d\sigma\, d\gamma,
\end{aligned}
\tag{4.1}
$$

where

$$
S_x = \sum_{i=1}^{n} y_i - n y_{(1)}, \qquad C_x = \frac{\sum_{j=1}^{m} z_j - m z_{(1)}}{\sum_{i=1}^{n} y_i - n y_{(1)}}
$$

and $y_{(1)}$ and $z_{(1)}$ are the first order statistics of the observations y_i and z_j respectively. Also, $\sigma \in R^+$ and $\gamma \in R^+$. The distribution of the slippage scale parameter γ is derived by integrating expression (4.1) over σ, which yields

$$
g^*(\gamma / \mathbf{y}, \mathbf{z}) = \frac{\Gamma(n+m)}{\Gamma(n)\Gamma(m)} \frac{C_x^m}{\left(1+\frac{C_x}{\gamma}\right)^{n+m}} \frac{d\gamma}{\gamma^{m+1}} =
$$

$$
= \frac{\Gamma\left(\frac{2m+2n}{2}\right)\left(\frac{2m}{2n}\right)^{\frac{2m}{2}}}{\Gamma\left(\frac{2m}{2}\right)\Gamma\left(\frac{2n}{2}\right)} \frac{\left(\frac{n}{m}\frac{C_x}{\gamma}\right)^{\frac{2m}{2}-1}}{\left[1+\frac{2m}{2n}\left(\frac{n}{m}\frac{C_x}{\gamma}\right)\right]^{\frac{2m+2n}{2}}}\ d\left(\frac{n}{m}\frac{C_x}{\gamma}\right)
$$

Thus, the random quantity $\frac{n}{m}\frac{C_x}{\gamma}$ is distributed according to an F distribution with 2m and 2n degrees of freedom, which gives the structural distribution of the slippage scale parameter γ.

5. LOCATION SHIFT AND SCALE CHANGE MODEL

This is the combination model, which can be expressed in structural form as

$$\mathbf{X} = \mathbf{\Theta} \bullet \mathbf{E}$$

where $\mathbf{X}$ is a 4x(n+m) response matrix, $\mathbf{\Theta}$ is a 4x4 transformation matrix, and $\mathbf{E}$ is a 4x(n+m) error matrix, such that

$$\mathbf{X} = \begin{bmatrix} \mathbf{1}' & \mathbf{0}' \\ \mathbf{0}' & \mathbf{1}' \\ \mathbf{y}' & \mathbf{0}' \\ \mathbf{0}' & \mathbf{z}' \end{bmatrix}, \quad \mathbf{\Theta} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \mu & 0 & \sigma & 0 \\ 0 & \mu+\lambda & 0 & \gamma\sigma \end{bmatrix}, \quad \text{and} \quad \mathbf{E} = \begin{bmatrix} \mathbf{1}' & \mathbf{0}' \\ \mathbf{0}' & \mathbf{1}' \\ \mathbf{e}' & \mathbf{0}' \\ \mathbf{0}' & \boldsymbol{\varepsilon}' \end{bmatrix}$$

with $\quad \mathbf{y}' = [y_1\ y_2 \ldots y_n]\,,\quad \mathbf{z}' = [z_1\ z_2 \ldots z_m]$

and $\quad \mathbf{e}' = [e_1\ e_2 \ldots e_n]\,,\quad \boldsymbol{\varepsilon}' = [\varepsilon_1\ \varepsilon_2 \ldots \varepsilon_m]\,.$

Structural analysis of this model yields the structural distribution of the parameters given the data (Armenakis 1988, pages 87-103) as

$$g^*\,(\mu, \lambda, \sigma, \gamma \,/\, \mathbf{y}, \mathbf{z})\, d\mu\, d\lambda\, d\sigma\, d\gamma = \frac{nm}{\Gamma(n-1) \qquad \Gamma(m-1)}\,\cdot$$

$$\cdot\ \exp\left\{-\frac{1}{\sigma}\left[n(y_{(1)}-\mu)+\frac{m}{\gamma}(z_{(1)}-\lambda-\mu)+S_X\left(1+\frac{C_X}{\gamma}\right)\right]\right\}\,\cdot \tag{5.1}$$

$$\cdot\ \frac{C_X^{m-1} \qquad S_X^{n+m-2}}{\gamma^{m+1} \qquad \sigma^{n+m+1}}\ d\mu\, d\lambda\, d\sigma\, d\gamma,$$

where $y_{(1)}$ and $z_{(1)}$ are the first order statistics of the observations y_i and z_j respectively,

$$S_X = \sum_{i=1}^{n} y_i - n y_{(1)}\,, \qquad \text{and} \qquad C_X = \frac{\sum_{j=1}^{m} z_j - m z_{(1)}}{\sum_{i=1}^{n} y_i - n y_{(1)}}$$

Also, $\mu<\mu_0 = \min\{y_{(1)}, z_{(1)}-\lambda\}$, $\lambda\in R$, $\sigma\in R^+$, and $\gamma\in R^+$. To find the distributions of the slippage parameters λ and γ, we need to integrate expression (5.1) with respect to σ and μ. Integrating over σ, we obtain that

$$g^*(\mu, \lambda, \gamma / \mathbf{y}, \mathbf{z})\, d\mu\, d\lambda\, d\gamma = \frac{n\cdot m\cdot \Gamma(n+m)\cdot C_X^{m-1}\cdot S_X^{n+m-2}}{\Gamma(n-1)\cdot\Gamma(m-1)\gamma^{m+1}} \cdot$$
$$\cdot \left[n(y_{(1)}-\mu)+\frac{m}{\gamma}(z_{(1)}-\lambda-\mu)+S_X\left(1+\frac{C_X}{\gamma}\right)\right]^{-(n+m)} d\mu\, d\lambda\, d\gamma \tag{5.2}$$

To integrate expression (5.2) over μ, $\mu<\mu_0$, it is necessary to distinguish between the cases $\mu_0 = y_{(1)}$ and $\mu_0 = z_{(1)}-\lambda$.

Case 1: Let $\mu_0 = y_{(1)}$, which also implies that $y_{(1)}<z_{(1)}-\lambda$ and therefore $\lambda<z_{(1)}-y_{(1)}$. Then, the structural distribution for the slippage parameters λ and γ is given by

$$g_1^*(\lambda / \mathbf{y}, \mathbf{z})\, d\lambda\, d\gamma = \frac{n\cdot m\cdot\Gamma(n+m-1)\cdot C_X^{m-1}\cdot S_X^{n+m-2}}{\Gamma(n-1)\cdot\Gamma(m-1)}\frac{\gamma^{n-1}}{(n\gamma+m)} \cdot$$
$$\cdot \left[-m\lambda+m(z_{(1)}-y_{(1)})+S_X(\gamma+C_X)\right]^{-(n+m-1)} d\lambda\, d\gamma, \tag{5.3}$$

where $-\infty<\lambda<z_{(1)}-y_{(1)}$ and $\gamma\in R^+$. Integration of expression (5.3) over λ yields

$$g_1^*(\gamma / \mathbf{y}, \mathbf{z})\, d\gamma = \frac{n\gamma}{n\gamma+m} \cdot g^*[f(\gamma) / \mathbf{y}, \mathbf{z}]\, df =$$
$$= \frac{n\gamma}{n\gamma+m}\cdot\frac{\Gamma\left[\frac{(2m-2)+(2n-2)}{2}\right]\left(\frac{2m-2}{2n-2}\right)^{\frac{2m-2}{2}}}{\Gamma\left(\frac{2m-2}{2}\right)\cdot\Gamma\left(\frac{2n-2}{2}\right)}\frac{f^{\frac{2m-2}{2}-1}}{\left(1+\frac{2m-2}{2n-2}f\right)^{\frac{(2m-2)+(2n-2)}{2}}}df \tag{5.4}$$

where $f = f(\gamma) = \dfrac{n-1}{m-1}\cdot\dfrac{C_X}{\gamma}$.

Case 2: In this case let $\mu_0 = z_{(1)}-\lambda$, which also implies that $z_{(1)}-\lambda<y_{(1)}$ and therefore $z_{(1)}-y_{(1)}<\lambda$. Then, the structural distribution for the slippage parameters λ and γ is obtained as

$$g_2^*(\lambda, \gamma / \mathbf{y}, \mathbf{z})\, d\lambda\, d\gamma = \frac{nm\Gamma(n+m-1)C_X^{m-1}S_X^{n+m-2}}{\Gamma(n-1)\cdot\Gamma(m-1)} \frac{\gamma^{n-1}}{(n\gamma+m)} \cdot$$

$$\cdot \left[n\gamma\lambda - n\gamma(z_{(1)} - y_{(1)}) + S_X(\gamma + C_X)\right]^{-(n+m-1)} \, d\lambda\, d\gamma \,, \tag{5.5}$$

where $z_{(1)}-y_{(1)}<\lambda<+\infty$ and $\gamma\in R^+$. Integration of expression (5.5) over λ gives that

$$g_2^*(\gamma / \mathbf{y}, \mathbf{z})\, d\gamma = \frac{m}{n\gamma+m} \cdot g^*[f(\gamma) / \mathbf{y}, \mathbf{z}]\, df =$$

$$= \frac{m}{n\gamma+m} \cdot \frac{\Gamma\left[\frac{(2m-2)+(2n-2)}{2}\right]\left(\frac{2m-2}{2n-2}\right)^{\frac{2m-2}{2}}}{\Gamma\left(\frac{2m-2}{2}\right)\cdot\Gamma\left(\frac{2n-2}{2}\right)} \frac{f^{\frac{2m-2}{2}-1}\, df}{\left(1+\frac{2m-2}{2n-2}f\right)^{\frac{(2m-2)+(2n-2)}{2}}} \tag{5.6}$$

where again $f = f(\gamma) = \dfrac{n-1}{m-1}\dfrac{C_X}{\gamma}$.

Using expressions (5.3) and (5.5) and then expressions (5.4) and (5.6), we have

$$\int_0^{+\infty}\int_{-\infty}^{+\infty} g^*(\lambda, \gamma / \mathbf{y}, \mathbf{z})\, d\lambda\, d\gamma =$$

$$= \int_0^{+\infty}\left[\int_{-\infty}^{z_{(1)}-y_{(1)}} g_1^*(\lambda, \gamma / \mathbf{y}, \mathbf{z})\, d\lambda + \int_{z_{(1)}-y_{(1)}}^{+\infty} g_2^*(\lambda, \gamma / \mathbf{y}, \mathbf{z})\, d\lambda\right] d\gamma =$$

$$= \int_0^{+\infty} \left[g_1^*(\gamma / \mathbf{y}, \mathbf{z}) + g_2^*(\gamma / \mathbf{y}, \mathbf{z})\right] d\gamma = 1,$$

where the fact that the random quantity f has an F distribution with 2m–2 and 2n–2 degrees of freedom was used. Therefore, the validity of the structural distribution for the parameters λ and γ is verified. Thus, the following results are obtained: the structural distribution for the location shift parameter λ is given by expressions (5.3) and (5.5) as a conditional distribution given γ. Also, the random quantity f has an F distribution with 2m–2 and 2n–2 degrees of freedom, which gives the structural distribution for the scale change parameter γ.

To construct interval estimates for the parameters λ and γ, the following procedure is suggested. First derive an interval estimate for the scale change parameter γ, using the fact that f has an F distribution with 2m–2 and 2n–2 degrees of freedom. If the hypothesis $\gamma = 1$ for the parameter γ can not be rejected, this implies that no slippage has occured with respect to scale. Then the value $\gamma = 1$ can be used in expressions (5.3) and (5.5) for further slippage testing with respect to location. Otherwise, if the hypothesis $\gamma = 1$ for the parameter γ is rejected, this implies that slippage with respect to scale has occured. Then the maximum likelihood estimator $\hat{\gamma}$ can be used in expressions (5.3) and (5.5) for further slippage testing with respect to location. The estimator $\hat{\gamma}$ is obtained from the maximum likelihood estimation method by setting

$$\frac{\partial \cdot \ln g^{*}[f(\gamma)/\mathbf{y},\mathbf{z}]}{\partial\gamma} = 0$$

where $g^{*}\,[f(\gamma)\,/\,\mathbf{y},\,\mathbf{z}]$ is the structural density of $f(\gamma)$. Then,

$$\hat{\gamma} = \frac{n}{m-2} C_X\,, \qquad m>2$$

The structural distribution of λ can now be used to construct 100(1-α)% interval estimates (λ_L, λ_U) for the parameter λ. Let

$$\gamma_0 = \begin{cases} 1, & \text{if} \quad H_0: \;\; \gamma=1 \quad \text{can not be rejected.} \\ \hat{\gamma}, & \text{if} \quad H_0: \;\; \gamma=1 \quad \text{is rejected.} \end{cases}$$

Consider the case where $\frac{\alpha}{2} \langle \frac{n\gamma_0}{n\gamma_0+m}$ and $\frac{\alpha}{2} \langle \frac{m}{n\gamma_0+m}$. Then the lower limit λ_L can be obtained from the integration of expression (5.3) as

$$\int_{-\infty}^{\lambda_L} g_1^{*}(\lambda,\gamma=\gamma_0\,/\,\mathbf{y},\mathbf{z})d\lambda = \frac{\alpha}{2}\,,$$

which gives

$$\lambda_L = z_{(1)} - y_{(1)} + \frac{S_X}{m}(\gamma_0 + C_X) - \frac{S_X}{m}\left[\frac{2\cdot n\cdot \gamma_0 \cdot \Gamma(n+m-2)\cdot C_X^{m-1}\cdot \gamma_0^{n-2}}{(n\gamma_0+m)\cdot \Gamma(n-1)\cdot \Gamma(m-1)\cdot \alpha}\right]^{\frac{1}{n+m-2}} \tag{5.7}$$

Similarly, the upper limit λ_U can be derived from the integration of expression (5.5) as

$$\int_{\lambda_U}^{+\infty} g_2^*(\lambda, \gamma = \gamma_0 / \mathbf{y}, \mathbf{z}) d\lambda = \frac{\alpha}{2},$$

which yields

$$\lambda_U = z_{(1)} - y_{(1)} - \frac{S_X}{n\gamma_0}(\gamma_0 + C_X) + \frac{S_X}{n\gamma_0}\left[\frac{2\cdot m\cdot \Gamma(n+m-2)\cdot C_X^{m-1}\cdot \gamma_0^{n-2}}{(n\gamma_0+m)\cdot \Gamma(n-1)\cdot \Gamma(m-1)\cdot \alpha}\right]^{\frac{1}{n+m-2}}$$

If $\frac{\alpha}{2} \langle \frac{n\gamma_0}{n\gamma_0+m}$ and $\frac{m}{n\gamma_0+m} \langle \frac{\alpha}{2}$, then the lower limit λ_L is again given by equation (5.7). Also, in this case the upper limit λ_U can be obtained from the integration of expression (5.3) as

$$\int_{-\infty}^{\lambda_U} g_1^*(\lambda, \gamma = \gamma_0 / \mathbf{y}, \mathbf{z}) d\lambda = 1 - \frac{\alpha}{2}$$

which yields

$$\lambda_U = z_{(1)} - y_{(1)} + \frac{S_X}{m}(\gamma_0 + C_X) - \frac{S_X}{m}\left[\frac{2\cdot n\cdot \gamma_0 \cdot \Gamma(n+m-2)\cdot C_X^{m-1}\cdot \gamma_0^{n-2}}{(n\gamma_0+m)\cdot \Gamma(n-1)\cdot \Gamma(m-1)\cdot (2-\alpha)}\right]^{\frac{1}{n+m-2}}$$

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